8.5 Partial Fractions

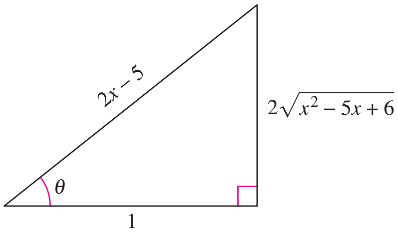
- **Understand the concept of partial fraction decomposition.**
- **Use partial fraction decomposition with linear factors to integrate rational functions.**
- **Use partial fraction decomposition with quadratic factors to integrate rational functions.**

Partial Fractions

This section examines a procedure for decomposing a rational function into simpler rational functions to which you can apply the basic integration formulas. This procedure is called the **method of partial fractions.** To see the benefit of the method of partial fractions, consider the integral

$$\int \frac{1}{x^2 - 5x + 6}\,dx.$$

To evaluate this integral *without* partial fractions, you can complete the square and use trigonometric substitution (see Figure 8.13) to obtain



$\sec\theta = 2x - 5$
Figure 8.13

$$\begin{aligned}
\int \frac{1}{x^2 - 5x + 6}\,dx &= \int \frac{dx}{(x - 5/2)^2 - (1/2)^2} && a = \tfrac{1}{2},\ x - \tfrac{5}{2} = \tfrac{1}{2}\sec\theta \\
&= \int \frac{(1/2)\sec\theta\tan\theta\,d\theta}{(1/4)\tan^2\theta} && dx = \tfrac{1}{2}\sec\theta\tan\theta\,d\theta \\
&= 2\int \csc\theta\,d\theta \\
&= 2\ln|\csc\theta - \cot\theta| + C \\
&= 2\ln\left|\frac{2x - 5}{2\sqrt{x^2 - 5x + 6}} - \frac{1}{2\sqrt{x^2 - 5x + 6}}\right| + C \\
&= 2\ln\left|\frac{x - 3}{\sqrt{x^2 - 5x + 6}}\right| + C \\
&= 2\ln\left|\frac{\sqrt{x - 3}}{\sqrt{x - 2}}\right| + C \\
&= \ln\left|\frac{x - 3}{x - 2}\right| + C \\
&= \ln|x - 3| - \ln|x - 2| + C.
\end{aligned}$$

Now, suppose you had observed that

$$\frac{1}{x^2 - 5x + 6} = \frac{1}{x - 3} - \frac{1}{x - 2}. \qquad \text{Partial fraction decomposition}$$

Then you could evaluate the integral, as shown.

$$\begin{aligned}
\int \frac{1}{x^2 - 5x + 6}\,dx &= \int \left(\frac{1}{x - 3} - \frac{1}{x - 2}\right) dx \\
&= \ln|x - 3| - \ln|x - 2| + C
\end{aligned}$$

This method is clearly preferable to trigonometric substitution. Its use, however, depends on the ability to factor the denominator, $x^2 - 5x + 6$, and to find the **partial fractions**

$$\frac{1}{x - 3} \quad \text{and} \quad -\frac{1}{x - 2}.$$

In this section, you will study techniques for finding partial fraction decompositions.

JOHN BERNOULLI (1667–1748)
The method of partial fractions was introduced by John Bernoulli, a Swiss mathematician who was instrumental in the early development of calculus. John Bernoulli was a professor at the University of Basel and taught many outstanding students, the most famous of whom was Leonhard Euler.
See LarsonCalculus.com to read more of this biography.

The Granger Collection

Recall from algebra that every polynomial with real coefficients can be factored into linear and irreducible quadratic factors.* For instance, the polynomial

$$x^5 + x^4 - x - 1$$

can be written as

$$\begin{aligned} x^5 + x^4 - x - 1 &= x^4(x + 1) - (x + 1) \\ &= (x^4 - 1)(x + 1) \\ &= (x^2 + 1)(x^2 - 1)(x + 1) \\ &= (x^2 + 1)(x + 1)(x - 1)(x + 1) \\ &= (x - 1)(x + 1)^2(x^2 + 1) \end{aligned}$$

where $(x - 1)$ is a linear factor, $(x + 1)^2$ is a repeated linear factor, and $(x^2 + 1)$ is an irreducible quadratic factor. Using this factorization, you can write the partial fraction decomposition of the rational expression

$$\frac{N(x)}{x^5 + x^4 - x - 1}$$

where $N(x)$ is a polynomial of degree less than 5, as shown.

$$\frac{N(x)}{(x - 1)(x + 1)^2(x^2 + 1)} = \frac{A}{x - 1} + \frac{B}{x + 1} + \frac{C}{(x + 1)^2} + \frac{Dx + E}{x^2 + 1}$$

REMARK In precalculus, you learned how to combine functions such as

$$\frac{1}{x - 2} + \frac{-1}{x + 3} = \frac{5}{(x - 2)(x + 3)}.$$

The method of partial fractions shows you how to reverse this process.

$$\frac{5}{(x - 2)(x + 3)} = \frac{?}{x - 2} + \frac{?}{x + 3}$$

Decomposition of $N(x)/D(x)$ into Partial Fractions

1. **Divide when improper:** When $N(x)/D(x)$ is an improper fraction (that is, when the degree of the numerator is greater than or equal to the degree of the denominator), divide the denominator into the numerator to obtain

$$\frac{N(x)}{D(x)} = (\text{a polynomial}) + \frac{N_1(x)}{D(x)}$$

   where the degree of $N_1(x)$ is less than the degree of $D(x)$. Then apply Steps 2, 3, and 4 to the proper rational expression $N_1(x)/D(x)$.

2. **Factor denominator:** Completely factor the denominator into factors of the form

$$(px + q)^m \quad \text{and} \quad (ax^2 + bx + c)^n$$

   where $ax^2 + bx + c$ is irreducible.

3. **Linear factors:** For each factor of the form $(px + q)^m$, the partial fraction decomposition must include the following sum of m fractions.

$$\frac{A_1}{(px + q)} + \frac{A_2}{(px + q)^2} + \cdots + \frac{A_m}{(px + q)^m}$$

4. **Quadratic factors:** For each factor of the form $(ax^2 + bx + c)^n$, the partial fraction decomposition must include the following sum of n fractions.

$$\frac{B_1x + C_1}{ax^2 + bx + c} + \frac{B_2x + C_2}{(ax^2 + bx + c)^2} + \cdots + \frac{B_nx + C_n}{(ax^2 + bx + c)^n}$$

* For a review of factorization techniques, see *Precalculus*, 9th edition, or *Precalculus: Real Mathematics*, *Real People*, 6th edition, both by Ron Larson (Boston, Massachusetts: Brooks/Cole, Cengage Learning, 2014 and 2012, respectively).

Linear Factors

Algebraic techniques for determining the constants in the numerators of a partial fraction decomposition with linear or repeated linear factors are shown in Examples 1 and 2.

EXAMPLE 1 Distinct Linear Factors

Write the partial fraction decomposition for

$$\frac{1}{x^2 - 5x + 6}.$$

Solution Because $x^2 - 5x + 6 = (x - 3)(x - 2)$, you should include one partial fraction for each factor and write

$$\frac{1}{x^2 - 5x + 6} = \frac{A}{x - 3} + \frac{B}{x - 2}$$

where A and B are to be determined. Multiplying this equation by the least common denominator $(x - 3)(x - 2)$ yields the **basic equation**

$$1 = A(x - 2) + B(x - 3). \quad \text{Basic equation}$$

Because this equation is to be true for all x, you can substitute any *convenient* values for x to obtain equations in A and B. The most convenient values are the ones that make particular factors equal to 0.

REMARK Note that the substitutions for x in Example 1 are chosen for their convenience in determining values for A and B; $x = 3$ is chosen to eliminate the term $B(x - 3)$, and $x = 2$ is chosen to eliminate the term $A(x - 2)$. The goal is to make *convenient* substitutions whenever possible.

To solve for A, let $x = 3$.

$$1 = A(3 - 2) + B(3 - 3) \quad \text{Let } x = 3 \text{ in basic equation.}$$
$$1 = A(1) + B(0)$$
$$1 = A$$

To solve for B, let $x = 2$.

$$1 = A(2 - 2) + B(2 - 3) \quad \text{Let } x = 2 \text{ in basic equation.}$$
$$1 = A(0) + B(-1)$$
$$-1 = B$$

So, the decomposition is

$$\frac{1}{x^2 - 5x + 6} = \frac{1}{x - 3} - \frac{1}{x - 2}$$

as shown at the beginning of this section.

FOR FURTHER INFORMATION To learn a different method for finding partial fraction decompositions, called the Heavyside Method, see the article "Calculus to Algebra Connections in Partial Fraction Decomposition" by Joseph Wiener and Will Watkins in *The AMATYC Review.*

Be sure you see that the method of partial fractions is practical only for integrals of rational functions whose denominators factor "nicely." For instance, when the denominator in Example 1 is changed to

$$x^2 - 5x + 5$$

its factorization as

$$x^2 - 5x + 5 = \left[x - \frac{5 + \sqrt{5}}{2}\right]\left[x - \frac{5 - \sqrt{5}}{2}\right]$$

would be too cumbersome to use with partial fractions. In such cases, you should use completing the square or a computer algebra system to perform the integration. When you do this, you should obtain

$$\int \frac{1}{x^2 - 5x + 5}\,dx = \frac{\sqrt{5}}{5}\ln\left|2x - \sqrt{5} - 5\right| - \frac{\sqrt{5}}{5}\ln\left|2x + \sqrt{5} - 5\right| + C.$$

FOR FURTHER INFORMATION For an alternative approach to using partial fractions, see the article "A Shortcut in Partial Fractions" by Xun-Cheng Huang in *The College Mathematics Journal*.

EXAMPLE 2 Repeated Linear Factors

Find $\displaystyle\int \frac{5x^2 + 20x + 6}{x^3 + 2x^2 + x}\,dx.$

Solution Because

$$x^3 + 2x^2 + x = x(x^2 + 2x + 1) = x(x + 1)^2$$

you should include one fraction for *each power* of x and $(x + 1)$ and write

$$\frac{5x^2 + 20x + 6}{x(x + 1)^2} = \frac{A}{x} + \frac{B}{x + 1} + \frac{C}{(x + 1)^2}.$$

Multiplying by the least common denominator $x(x + 1)^2$ yields the *basic equation*

$$5x^2 + 20x + 6 = A(x + 1)^2 + Bx(x + 1) + Cx. \qquad \text{Basic equation}$$

To solve for A, let $x = 0$. This eliminates the B and C terms and yields

$$6 = A(1) + 0 + 0$$
$$6 = A.$$

To solve for C, let $x = -1$. This eliminates the A and B terms and yields

$$5 - 20 + 6 = 0 + 0 - C$$
$$9 = C.$$

The most convenient choices for x have been used, so to find the value of B, you can use *any other value* of x along with the calculated values of A and C. Using $x = 1$, $A = 6$, and $C = 9$ produces

$$5 + 20 + 6 = A(4) + B(2) + C$$
$$31 = 6(4) + 2B + 9$$
$$-2 = 2B$$
$$-1 = B.$$

So, it follows that

$$\begin{aligned}\int \frac{5x^2 + 20x + 6}{x(x + 1)^2}\,dx &= \int \left(\frac{6}{x} - \frac{1}{x + 1} + \frac{9}{(x + 1)^2}\right) dx \\ &= 6 \ln|x| - \ln|x + 1| + 9\frac{(x + 1)^{-1}}{-1} + C \\ &= \ln\left|\frac{x^6}{x + 1}\right| - \frac{9}{x + 1} + C.\end{aligned}$$

Try checking this result by differentiating. Include algebra in your check, simplifying the derivative until you have obtained the original integrand.

It is necessary to make as many substitutions for x as there are unknowns $(A, B, C, \ldots)$ to be determined. For instance, in Example 2, three substitutions $(x = 0, x = -1$, and $x = 1)$ were made to solve for A, B, and C.

TECHNOLOGY Most computer algebra systems, such as *Maple, Mathematica,* and the *TI-nSpire,* can be used to convert a rational function to its partial fraction decomposition. For instance, using *Mathematica,* you obtain the following.

Apart $[(5 * x \wedge 2 + 20 * x + 6)/(x * (x + 1) \wedge 2), x]$

$$\frac{6}{x} + \frac{9}{(1 + x)^2} - \frac{1}{1 + x}$$

Quadratic Factors

When using the method of partial fractions with *linear* factors, a convenient choice of x immediately yields a value for one of the coefficients. With *quadratic* factors, a system of linear equations usually has to be solved, regardless of the choice of x.

EXAMPLE 3 Distinct Linear and Quadratic Factors

See LarsonCalculus.com for an interactive version of this type of example.

Find $\displaystyle\int \frac{2x^3 - 4x - 8}{(x^2 - x)(x^2 + 4)}\,dx.$

Solution Because

$$(x^2 - x)(x^2 + 4) = x(x - 1)(x^2 + 4)$$

you should include one partial fraction for each factor and write

$$\frac{2x^3 - 4x - 8}{x(x - 1)(x^2 + 4)} = \frac{A}{x} + \frac{B}{x - 1} + \frac{Cx + D}{x^2 + 4}.$$

Multiplying by the least common denominator

$$x(x - 1)(x^2 + 4)$$

yields the *basic equation*

$$2x^3 - 4x - 8 = A(x - 1)(x^2 + 4) + Bx(x^2 + 4) + (Cx + D)(x)(x - 1).$$

To solve for A, let $x = 0$ and obtain

$$-8 = A(-1)(4) + 0 + 0$$
$$2 = A.$$

To solve for B, let $x = 1$ and obtain

$$-10 = 0 + B(5) + 0$$
$$-2 = B.$$

At this point, C and D are yet to be determined. You can find these remaining constants by choosing two other values for x and solving the resulting system of linear equations. Using $x = -1$, $A = 2$, and $B = -2$, you can write

$$-6 = (2)(-2)(5) + (-2)(-1)(5) + (-C + D)(-1)(-2)$$
$$2 = -C + D.$$

For $x = 2$, you have

$$0 = (2)(1)(8) + (-2)(2)(8) + (2C + D)(2)(1)$$
$$8 = 2C + D.$$

Solving the linear system by subtracting the first equation from the second

$$-C + D = 2$$
$$2C + D = 8$$

yields $C = 2$. Consequently, $D = 4$, and it follows that

$$\int \frac{2x^3 - 4x - 8}{x(x - 1)(x^2 + 4)}\,dx = \int \left(\frac{2}{x} - \frac{2}{x - 1} + \frac{2x}{x^2 + 4} + \frac{4}{x^2 + 4}\right) dx$$

$$= 2\ln|x| - 2\ln|x - 1| + \ln(x^2 + 4) + 2\arctan\frac{x}{2} + C.$$

In Examples 1, 2, and 3, the solution of the basic equation began with substituting values of x that made the linear factors equal to 0. This method works well when the partial fraction decomposition involves linear factors. When the decomposition involves only quadratic factors, however, an alternative procedure is often more convenient. For instance, try writing the right side of the basic equation in polynomial form and *equating the coefficients* of like terms. This method is shown in Example 4.

EXAMPLE 4 Repeated Quadratic Factors

Find $\displaystyle\int \frac{8x^3 + 13x}{(x^2 + 2)^2}\,dx.$

Solution Include one partial fraction for each power of $(x^2 + 2)$ and write

$$\frac{8x^3 + 13x}{(x^2 + 2)^2} = \frac{Ax + B}{x^2 + 2} + \frac{Cx + D}{(x^2 + 2)^2}.$$

Multiplying by the least common denominator $(x^2 + 2)^2$ yields the *basic equation*

$$8x^3 + 13x = (Ax + B)(x^2 + 2) + Cx + D.$$

Expanding the basic equation and collecting like terms produces

$$8x^3 + 13x = Ax^3 + 2Ax + Bx^2 + 2B + Cx + D$$
$$8x^3 + 13x = Ax^3 + Bx^2 + (2A + C)x + (2B + D).$$

Now, you can equate the coefficients of like terms on opposite sides of the equation.

$$8 = A \qquad 0 = 2B + D$$

$$8x^3 + 0x^2 + 13x + 0 = Ax^3 + Bx^2 + (2A + C)x + (2B + D)$$

$$0 = B$$

$$13 = 2A + C$$

Using the known values $A = 8$ and $B = 0$, you can write

$$13 = 2A + C \;\Rightarrow\; 13 = 2(8) + C \;\Rightarrow\; -3 = C$$
$$0 = 2B + D \;\Rightarrow\; 0 = 2(0) + D \;\Rightarrow\; 0 = D.$$

Finally, you can conclude that

$$\int \frac{8x^3 + 13x}{(x^2 + 2)^2}\,dx = \int \left(\frac{8x}{x^2 + 2} + \frac{-3x}{(x^2 + 2)^2}\right) dx$$
$$= 4\ln(x^2 + 2) + \frac{3}{2(x^2 + 2)} + C.$$

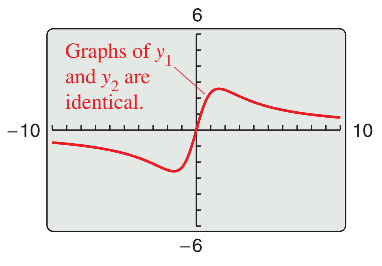
TECHNOLOGY You can use a graphing utility to confirm the decomposition found in Example 4. To do this, graph

$$y_1 = \frac{8x^3 + 13x}{(x^2 + 2)^2}$$

and

$$y_2 = \frac{8x}{x^2 + 2} + \frac{-3x}{(x^2 + 2)^2}$$

in the same viewing window. The graphs should be identical, as shown at the right.



When integrating rational expressions, keep in mind that for *improper* rational expressions such as

$$\frac{N(x)}{D(x)} = \frac{2x^3 + x^2 - 7x + 7}{x^2 + x - 2}$$

you must first divide to obtain

$$\frac{N(x)}{D(x)} = 2x - 1 + \frac{-2x + 5}{x^2 + x - 2}.$$

The proper rational expression is then decomposed into its partial fractions by the usual methods.

Here are some guidelines for solving the basic equation that is obtained in a partial fraction decomposition.

GUIDELINES FOR SOLVING THE BASIC EQUATION

Linear Factors

1. Substitute the roots of the distinct linear factors in the basic equation.
2. For repeated linear factors, use the coefficients determined in the first guideline to rewrite the basic equation. Then substitute other convenient values of x and solve for the remaining coefficients.

Quadratic Factors

1. Expand the basic equation.
2. Collect terms according to powers of x.
3. Equate the coefficients of like powers to obtain a system of linear equations involving A, B, C, and so on.
4. Solve the system of linear equations.

FOR FURTHER INFORMATION To read about another method of evaluating integrals of rational functions, see the article "Alternate Approach to Partial Fractions to Evaluate Integrals of Rational Functions" by N. R. Nandakumar and Michael J. Bossé in *The Pi Mu Epsilon Journal.* To view this article, go to *MathArticles.com.*

Before concluding this section, here are a few things you should remember. First, it is not necessary to use the partial fractions technique on all rational functions. For instance, the following integral is evaluated more easily by the Log Rule.

$$\int \frac{x^2 + 1}{x^3 + 3x - 4}\,dx = \frac{1}{3}\int \frac{3x^2 + 3}{x^3 + 3x - 4}\,dx$$
$$= \frac{1}{3}\ln|x^3 + 3x - 4| + C$$

Second, when the integrand is not in reduced form, reducing it may eliminate the need for partial fractions, as shown in the following integral.

$$\int \frac{x^2 - x - 2}{x^3 - 2x - 4}\,dx = \int \frac{(x + 1)(x - 2)}{(x - 2)(x^2 + 2x + 2)}\,dx$$
$$= \int \frac{x + 1}{x^2 + 2x + 2}\,dx$$
$$= \frac{1}{2}\ln|x^2 + 2x + 2| + C$$

Finally, partial fractions can be used with some quotients involving transcendental functions. For instance, the substitution $u = \sin x$ allows you to write

$$\int \frac{\cos x}{\sin x(\sin x - 1)}\,dx = \int \frac{du}{u(u - 1)}. \qquad u = \sin x,\ du = \cos x\,dx$$

8.5 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Partial Fraction Decomposition **In Exercises 1–4, write the form of the partial fraction decomposition of the rational expression. Do not solve for the constants.**

1. $\dfrac{4}{x^2 - 8x}$

2. $\dfrac{2x^2 + 1}{(x - 3)^3}$

3. $\dfrac{2x - 3}{x^3 + 10x}$

4. $\dfrac{2x - 1}{x(x^2 + 1)^2}$

Using Partial Fractions **In Exercises 5–22, use partial fractions to find the indefinite integral.**

5. $\displaystyle\int \frac{1}{x^2 - 9}\,dx$

6. $\displaystyle\int \frac{2}{9x^2 - 1}\,dx$

7. $\displaystyle\int \frac{5}{x^2 + 3x - 4}\,dx$

8. $\displaystyle\int \frac{3 - x}{3x^2 - 2x - 1}\,dx$

9. $\displaystyle\int \frac{x^2 + 12x + 12}{x^3 - 4x}\,dx$

10. $\displaystyle\int \frac{x^3 - x + 3}{x^2 + x - 2}\,dx$

11. $\displaystyle\int \frac{2x^3 - 4x^2 - 15x + 5}{x^2 - 2x - 8}\,dx$

12. $\displaystyle\int \frac{x + 2}{x^2 + 5x}\,dx$

13. $\displaystyle\int \frac{4x^2 + 2x - 1}{x^3 + x^2}\,dx$

14. $\displaystyle\int \frac{5x - 2}{(x - 2)^2}\,dx$

15. $\displaystyle\int \frac{x^2 + 3x - 4}{x^3 - 4x^2 + 4x}\,dx$

16. $\displaystyle\int \frac{8x}{x^3 + x^2 - x - 1}\,dx$

17. $\displaystyle\int \frac{x^2 - 1}{x^3 + x}\,dx$

18. $\displaystyle\int \frac{6x}{x^3 - 8}\,dx$

19. $\displaystyle\int \frac{x^2}{x^4 - 2x^2 - 8}\,dx$

20. $\displaystyle\int \frac{x}{16x^4 - 1}\,dx$

21. $\displaystyle\int \frac{x^2 + 5}{x^3 - x^2 + x + 3}\,dx$

22. $\displaystyle\int \frac{x^2 + 6x + 4}{x^4 + 8x^2 + 16}\,dx$

Evaluating a Definite Integral **In Exercises 23–26, evaluate the definite integral. Use a graphing utility to verify your result.**

23. $\displaystyle\int_0^2 \frac{3}{4x^2 + 5x + 1}\,dx$

24. $\displaystyle\int_1^5 \frac{x - 1}{x^2(x + 1)}\,dx$

25. $\displaystyle\int_1^2 \frac{x + 1}{x(x^2 + 1)}\,dx$

26. $\displaystyle\int_0^1 \frac{x^2 - x}{x^2 + x + 1}\,dx$

Finding an Indefinite Integral **In Exercises 27–34, use substitution and partial fractions to find the indefinite integral.**

27. $\displaystyle\int \frac{\sin x}{\cos x + \cos^2 x}\,dx$

28. $\displaystyle\int \frac{5\cos x}{\sin^2 x + 3\sin x - 4}\,dx$

29. $\displaystyle\int \frac{\sec^2 x}{\tan^2 x + 5\tan x + 6}\,dx$

30. $\displaystyle\int \frac{\sec^2 x}{\tan x(\tan x + 1)}\,dx$

31. $\displaystyle\int \frac{e^x}{(e^x - 1)(e^x + 4)}\,dx$

32. $\displaystyle\int \frac{e^x}{(e^{2x} + 1)(e^x - 1)}\,dx$

33. $\displaystyle\int \frac{\sqrt{x}}{x - 4}\,dx$

34. $\displaystyle\int \frac{1}{\sqrt{x} - \sqrt[3]{x}}\,dx$

Verifying a Formula **In Exercises 35–38, use the method of partial fractions to verify the integration formula.**

35. $\displaystyle\int \frac{1}{x(a + bx)}\,dx = \frac{1}{a}\ln\left|\frac{x}{a + bx}\right| + C$

36. $\displaystyle\int \frac{1}{a^2 - x^2}\,dx = \frac{1}{2a}\ln\left|\frac{a + x}{a - x}\right| + C$

37. $\displaystyle\int \frac{x}{(a + bx)^2}\,dx = \frac{1}{b^2}\left(\frac{a}{a + bx} + \ln|a + bx|\right) + C$

38. $\displaystyle\int \frac{1}{x^2(a + bx)}\,dx = -\frac{1}{ax} - \frac{b}{a^2}\ln\left|\frac{x}{a + bx}\right| + C$

WRITING ABOUT CONCEPTS

39. **Using Partial Fractions** What is the first step when integrating $\displaystyle\int \frac{x^3}{x - 5}\,dx$? Explain.

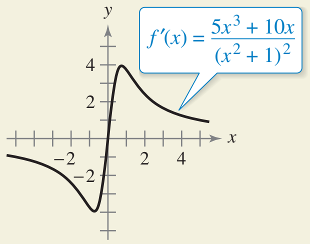
40. **Decomposition** Describe the decomposition of the proper rational function $N(x)/D(x)$ (a) for $D(x) = (px + q)^m$ and (b) for $D(x) = (ax^2 + bx + c)^n$ where $ax^2 + bx + c$ is irreducible. Explain why you chose that method.

41. **Choosing a Method** State the method you would use to evaluate each integral. Explain why you chose that method. Do not integrate.

(a) $\displaystyle\int \frac{x + 1}{x^2 + 2x - 8}\,dx$ (b) $\displaystyle\int \frac{7x + 4}{x^2 + 2x - 8}\,dx$

(c) $\displaystyle\int \frac{4}{x^2 + 2x + 5}\,dx$

42. **HOW DO YOU SEE IT?** Use the graph of f' shown in the figure to answer the following.



(a) Is $f(3) - f(2) > 0$? Explain.

(b) Which is greater, the area under the graph of f' from 1 to 2, or the area under the graph of f' from 3 to 4?

43. **Area** Find the area of the region bounded by the graphs of $y = 12/(x^2 + 5x + 6)$, $y = 0$, $x = 0$, and $x = 1$.

44. **Area** Find the area of the region bounded by the graphs of $y = 7/(16 - x^2)$ and $y = 1$.

45. Modeling Data The predicted cost C (in hundreds of thousands of dollars) for a company to remove $p\%$ of a chemical from its waste water is shown in the table.

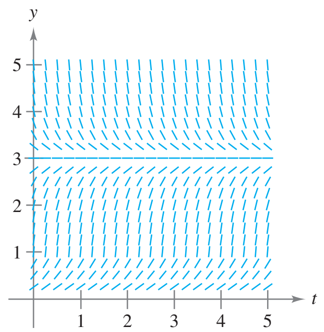
P	0	10	20	30	40	50	60	70	80	90
C	0	0.7	1.0	1.3	1.7	2.0	2.7	3.6	5.5	11.2

A model for the data is given by $C = \dfrac{124p}{(10 + p)(100 - p)}$ for $0 \le p < 100$. Use the model to find the average cost of removing between 75% and 80% of the chemical.

46. Logistic Growth In Chapter 6, the exponential growth equation was derived from the assumption that the rate of growth was proportional to the existing quantity. In practice, there often exists some upper limit L past which growth cannot occur. In such cases, you assume the rate of growth to be proportional not only to the existing quantity, but also to the difference between the existing quantity y and the upper limit L. That is, $dy/dt = ky(L - y)$. In integral form, you can write this relationship as

$$\int \frac{dy}{y(L - y)} = \int k\,dt.$$

(a) A slope field for the differential equation $dy/dt = y(3 - y)$ is shown. Draw a possible solution to the differential equation when $y(0) = 5$, and another when $y(0) = \frac{1}{2}$. To print an enlarged copy of the graph, go to *MathGraphs.com.*



(b) Where $y(0)$ is greater than 3, what is the sign of the slope of the solution?

(c) For $y > 0$, find $\lim_{t \to \infty} y(t)$.

(d) Evaluate the two given integrals and solve for y as a function of t, where y_0 is the initial quantity.

(e) Use the result of part (d) to find and graph the solutions in part (a). Use a graphing utility to graph the solutions and compare the results with the solutions in part (a).

(f) The graph of the function y is a **logistic curve.** Show that the rate of growth is maximum at the point of inflection, and that this occurs when $y = L/2$.

47. Volume and Centroid Consider the region bounded by the graphs of $y = 2x/(x^2 + 1)$, $y = 0$, $x = 0$, and $x = 3$. Find the volume of the solid generated by revolving the region about the x-axis. Find the centroid of the region.

48. Volume Consider the region bounded by the graph of

$$y^2 = \frac{(2 - x)^2}{(1 + x)^2}$$

on the interval $[0, 1]$. Find the volume of the solid generated by revolving this region about the x-axis.

49. Epidemic Model A single infected individual enters a community of n susceptible individuals. Let x be the number of newly infected individuals at time t. The common epidemic model assumes that the disease spreads at a rate proportional to the product of the total number infected and the number not yet infected. So, $dx/dt = k(x + 1)(n - x)$ and you obtain

$$\int \frac{1}{(x + 1)(n - x)}\,dx = \int k\,dt.$$

Solve for x as a function of t.

50. Chemical Reaction

In a chemical reaction, one unit of compound Y and one unit of compound Z are converted into a single unit of compound X. Let x be the amount of compound X formed. The rate of formation of X is proportional to the product of the amounts of unconverted compounds Y and Z. So, $dx/dt = k(y_0 - x)(z_0 - x)$, where y_0 and z_0 are the initial amounts of compounds Y and Z. From this equation, you obtain

$$\int \frac{1}{(y_0 - x)(z_0 - x)}\,dx = \int k\,dt.$$

(a) Perform the two integrations and solve for x in terms of t.

(b) Use the result of part (a) to find x as $t \to \infty$ for (1) $y_0 < z_0$, (2) $y_0 > z_0$, and (3) $y_0 = z_0$.

51. Using Two Methods Evaluate

$$\int_0^1 \frac{x}{1 + x^4}\,dx$$

in two different ways, one of which is partial fractions.

PUTNAM EXAM CHALLENGE

52. Prove $\dfrac{22}{7} - \pi = \displaystyle\int_0^1 \frac{x^4(1 - x)^4}{1 + x^2}\,dx.$

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